Find $\int \frac{1}{x^3+1} dx$

We know that $(x+1)$ is a root of x^3+1, since $(-1)^3+1=0$

By polynomial long division, we find that $x^3+1=(x+1)(x^2-x+1)$

Note that (x^2-x+1) is an irreducible quadratic since $b^2-4ac=1-4*1=-4<0$

We therefore use the partial fractions decomposition: $\frac{1}{x^3+1}=\frac{Ax+B}{x^2-x+1}+\frac{C}{x+1}$

By multiplying this equation through by the denominator x^3+1,
we get $1=(Ax+B)(x+1)+C(x^2-x+1)$

There a a number of ways to use this equation to find A, B, and C. Since this equation has to be true for all x, it is true at $x=-1$, yielding: $1=0+C(1+1+1)$, which implies that $C=\frac{1}{3}$.

Next, we multiply out the large equation: $1=Ax^2+Ax+Bx+B+Cx^2-Cx+C=(A+C)x^2+(A+B-C)x+(B+C)$. This yields the following equations relating the coefficients of x^2, x and the constant terms on the right and left of this equation:
$0=A+C$
$0=A+B-C$
$1=B+C$

Since we have compute C to be $\frac{1}{3}$, the coefficients of x^2 immediately yield that $A=-C=-\frac{1}{3}$, whereas the coefficients of x^2 yield that $B=C-A=C-(-C)=2C=\frac{2}{3}$

To clarify: $A=-\frac{1}{3}; B=\frac{2}{3}; C=\frac{1}{3}$

We now rewrite our integral using this partial fractions decomposition:

$$\int \frac{1}{x^3+1}\, dx = \frac{1}{3}\int \frac{-x+2}{x^2-x+1}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx$$
$$= \tfrac{1}{3}(-\tfrac{1}{2}) \int \frac{2x-4}{x^2-x+1}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx$$

Note: We see that the only way to solve the first integral (linear over quadratic) is with a u-substitution, but the constant -4 in the numerator will not match up with the desired substitution. We remove this constant before continuing:

$$= -\tfrac{1}{6}\int \frac{(2x-1)-3}{x^2-x+1}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx$$
$$= -\tfrac{1}{6}\int \frac{(2x-1)}{x^2-x+1}\, dx - \frac{1}{6}\int \frac{(-3)}{x^2-x+1}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx$$

Let $u = x^2 - x + 1$, then $du = (2x-1)dx$, simplifying the first integral. The second integral is type constant/ quadratic, so we must complete the square for the denominator of the second integral to get a form $\int \frac{1}{u^2+a^2}\, du = \frac{1}{a}\tan^{-1}(\frac{u}{a})$

$$= -\tfrac{1}{6}\int \frac{du}{u}\, dx + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\, dx + \frac{1}{3}\int \frac{1}{x+1}\, dx$$

$$= -\tfrac{1}{6}\ln|x^2-x+1| + \tfrac{1}{2} * \tfrac{2}{\sqrt{3}}\tan^{-1}(\tfrac{2}{\sqrt{3}} * (x-\tfrac{1}{2})) + \tfrac{1}{3}\int \frac{1}{x+1}\, dx$$

$$=\tfrac{1}{6}(-\ln|x^2-x+1| + 2\sqrt{3}\tan^{-1}(\tfrac{2x-1}{\sqrt{3}}) + 2\ln|x+1| + c)$$

Some key points from this example:

$\frac{linear}{quadratic}$ use u-substitution. Force the correct coefficient of x and take out (separate) any unwanted constants.

$\frac{constant}{quadratic}$ use a u-substitution with $\frac{1}{a}\tan^{-1}(\frac{u}{a})$; it may be necessary to complete the square first

$\frac{constant}{linear}$ use a u-substitution to get a $ln|u|$

This should cover all your bases when you get to the end of a partial fractions problem. I wish you better luck with your algebra than I had!

Happy Integrating,
Marisa